Problem 8 solved with a Green's function

Thundercloud above conducting ground

Part IB Electromagnetism, Michaelmas Term 2026

Deepak Venkateshvaran

dv246@cam.ac.uk

Questions and corrections are welcome by email.

Problem 8 (A) (as set on the sheet)

The electrical system of a typical thundercloud can be represented by a vertical dipole consisting of a charge of $+40\,\mathrm{C}$ at a height of $10\,\mathrm{km}$ and a charge of $-40\,\mathrm{C}$ vertically below it at a height of $6\,\mathrm{km}$. What is the electric field at the ground immediately below the thundercloud and at what distance from there is the field at the ground zero? *Hint: treat the ground as a perfect conductor.*

What you need to know before starting

- **Superposition:** potentials of separate charges add.
- **Uniqueness theorem:** in a region bounded by conductors, a potential that obeys Poisson's equation $\nabla^2\phi = -\rho_c/\varepsilon_0$ (with ρ_c the charge density and ε_0 the permittivity of free space) and has the right values on the conductors is the only possible answer.
- **Method of images for a plane:** a charge Q at height h above an earthed conducting plane, together with an "image" charge $-Q$ at depth h below it, gives zero potential on the plane. By uniqueness, the pair gives the correct field above the plane.
- **Point charges as a density:** a point charge Q at $\mathbf{r}_Q$ has $\rho_c = Q\,\delta^3(\mathbf{r}-\mathbf{r}_Q)$. The delta function δ^3 is zero except at $\mathbf{r}_Q$, and $\int f(\mathbf{r}')\,\delta^3(\mathbf{r}'-\mathbf{r}_Q)\,\mathrm{d}V' = f(\mathbf{r}_Q)$ for any function f.
- **Field from potential:** $\mathbf{E} = -\nabla\phi$, so the vertical field is $E_z = -\partial\phi/\partial z$.
- **Charge on a conductor:** $\sigma = \varepsilon_0 E_n$, with E_n the field just outside, pointing away from the conductor (Problem 6(b)).

Final answer

Directly below the cloud $E_z = 12.8\,\mathrm{kV\,m^{-1}}$, i.e. $12.8\,\mathrm{kV\,m^{-1}}$ pointing *upwards*. The ground field is zero on a circle of radius $11.0\,\mathrm{km}$ centred on the point below the cloud.

The idea: what a Green's function is

The **Green's function** $G(\mathbf{r},\mathbf{r}')$ is the potential at a point $\mathbf{r}$ (the *field point*) produced by a *unit* point charge at $\mathbf{r}'$ (the *source point*), with any conductors earthed. It includes the effect of the charge that the unit charge induces on the conductors.

Once G is known, the potential of *any* charge distribution near the same conductors follows from superposition. Cut the charge into small pieces; the piece at $\mathbf{r}'$ holds charge $\rho_c(\mathbf{r}')\,\mathrm{d}V'$ and contributes $G(\mathbf{r},\mathbf{r}')\,\rho_c(\mathbf{r}')\,\mathrm{d}V'$. Adding all the pieces,

$$\phi(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')\,\rho_c(\mathbf{r}')\,\mathrm{d}V'.$$

Every piece gives zero potential on the earthed conductors, so the sum does too, and by uniqueness this is the answer.

Properties of a Green's function

1. **It is the potential of a unit point charge.** So it obeys Poisson's equation with a point source,
$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\frac{1}{\varepsilon_0}\,\delta^3(\mathbf{r} - \mathbf{r}'),$$
where ∇^2 acts on the field point $\mathbf{r}$. Away from the source point it obeys Laplace's equation, $\nabla^2 G = 0$. Its units are $\mathrm{V\,C^{-1}}$.
2. **It vanishes on earthed conductors:** $G = 0$ whenever $\mathbf{r}$ is on an earthed surface.
3. **Uniqueness.** For a given set of earthed conductors there is exactly one function with properties 1 and 2. So a G found by *any* method, including an inspired guess such as the method of images, is correct as long as it passes both checks.
4. **Superposition.** For any charge density ρ_c in the same region,
$$\phi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')\,\rho_c(\mathbf{r}')\,\mathrm{d}V'.$$
5. **It depends only on the conductors, not on the charges.** It is found once for a given geometry and then reused for any charge distribution.
6. **Reciprocity:** $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$. A unit charge at A produces the same potential at B as a unit charge at B produces at A, even with earthed conductors present. For the earthed ground this is easy to check from Step 1: swapping $\mathbf{r}$ and $\mathbf{r}'$ leaves unchanged both the distance between them and the distance from each to the mirror image of the other.
7. **Free space** (no conductors): $G_0 = 1/(4\pi\varepsilon_0|\mathbf{r} - \mathbf{r}'|)$, which is Coulomb's law for a unit charge.
8. **Effect of the conductors.** The charge induced on an earthed conductor has the opposite sign to the unit charge, so it lowers the potential: in the region, $0 < G < G_0$. For the earthed ground, the induced charge totals exactly -1, because every field line from the unit charge ends on the ground (shown in "Going further", item (i)).

A note on conventions. Here G is the potential of a unit charge. Some textbooks instead define a Green's function by $\nabla^2 \mathcal{G} = \delta^3(\mathbf{r} - \mathbf{r}')$, without ε_0 and with the opposite sign, so that $\mathcal{G} = -\varepsilon_0 G$. The physics is the same; check which definition a book is using.

For this problem the conductor is the ground, the same for any cloud. So the strategy is: find G for the earthed ground once (Step 1), then add up the cloud's charges (Steps 2 to 5).

Worked solution

Step 0: set-up

Take the ground as the earthed conducting plane $z = 0$, with z measured upwards, and put the cloud on the z axis:

$$Q_1 = 40\,\mathrm{C} \text{ at height } h_1 = 10\,\mathrm{km}, \qquad Q_2 = -40\,\mathrm{C} \text{ at height } h_2 = 6\,\mathrm{km}.$$

By symmetry about the vertical axis, everything depends only on the height z and on the horizontal distance s from the axis (Figure 1).

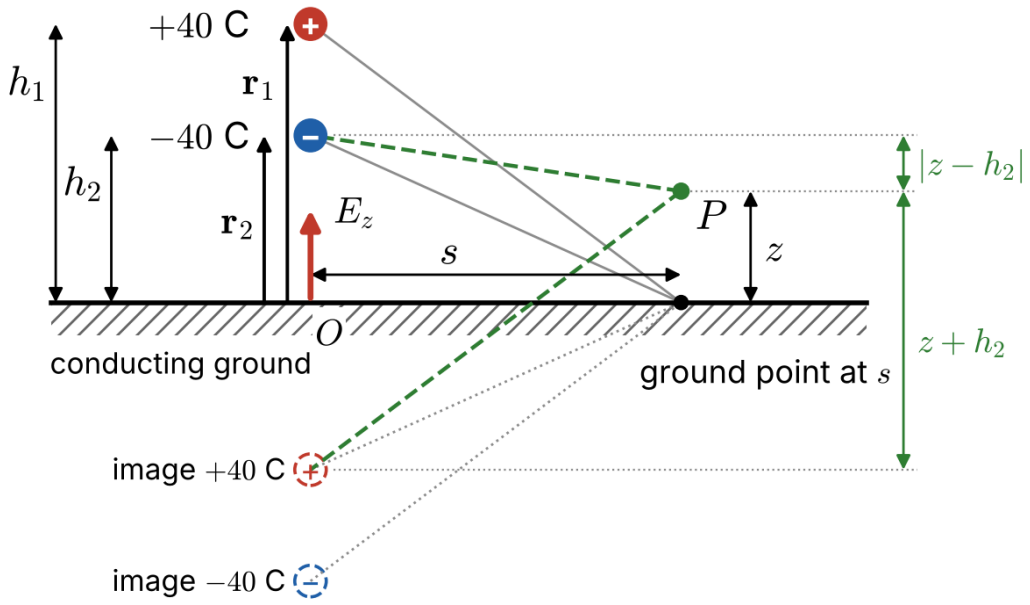


Figure 1: The cloud charges at heights h_1 and h_2 above the conducting ground, their images below it, and a ground point at horizontal distance s from the point below the cloud. O is the origin, on the ground directly below the cloud, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the two cloud charges measured from O (they lie along the vertical axis; they are drawn just beside it for clarity). P is a general field point at the same horizontal distance s and at height z. The green dashed lines join P to the $-40\,\mathrm{C}$ charge and to that charge's image: their vertical separations from P have lengths $|z-h_2|$ and $z+h_2$, which give the distances used in Step 1 (with $h=h_2$).

Step 1: the Green's function for the earthed ground

A unit charge on the axis at height h, together with its image -1 at depth h, gives the potential at a point at height z and horizontal distance s:

$$G(s,z;h)=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{s^2+(z-h)^2}}-\frac{1}{\sqrt{s^2+(z+h)^2}}\right].$$

Each distance comes from Pythagoras (Figure 1): the horizontal separation is s, and the vertical separation is the difference in heights, $z-h$ from the charge at height h and $z-(-h)=z+h$ from the image at height $-h$. The first term is the distance to the charge; the second is the distance to its image. The vertical separation is squared, so its sign does not matter: when P is below the charge (as in Figure 1), $z-h$ is negative, but $(z-h)^2=(h-z)^2$ is the square of the actual gap. At ground level ($z=0$) the two distances are equal, so $G=0$: the ground is earthed, as required. We find this once, for *any* height h.

Step 2: the potential of the cloud

The cloud's charge density is two point charges, $\rho_\mathrm{c}=Q_1\,\delta^3(\mathbf{r}-\mathbf{r}_1)+Q_2\,\delta^3(\mathbf{r}-\mathbf{r}_2)$, where $\mathbf{r}_1=(0,0,h_1)$ and $\mathbf{r}_2=(0,0,h_2)$ are the positions of the charges measured from O (Figure 1). Putting this into $\phi=\int G\,\rho_\mathrm{c}\,\mathrm{d}V'$, each delta function simply picks out G at that charge's position:

$$\phi(s,z)=Q_1\,G(s,z;h_1)+Q_2\,G(s,z;h_2).$$

In words: the potential is (charge) × (potential of a unit charge at that height), added over the two charges. This is the whole Green's function idea in one line.

Step 3: the field at the ground from a single unit charge

The vertical field is $E_z = -\partial\phi/\partial z$, so we need $\partial G/\partial z$ at ground level. Differentiate each term with respect to z (the field point's height; h is held fixed), using $\frac{\partial}{\partial z}\big(s^2+(z\mp h)^2\big)^{-1/2} = -(z\mp h)\big(s^2+(z\mp h)^2\big)^{-3/2}$:

$$\frac{\partial G}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[-\frac{z-h}{\big(s^2+(z-h)^2\big)^{3/2}} + \frac{z+h}{\big(s^2+(z+h)^2\big)^{3/2}}\right].$$

Now set $z=0$. Both denominators become $(s^2+h^2)^{3/2}$, and the numerators become $+h$ and $+h$:

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = \frac{1}{4\pi\varepsilon_0}\,\frac{2h}{(s^2+h^2)^{3/2}}.$$

So the vertical field at the ground due to a unit charge at height h is

$$\boxed{g(s;h) \equiv -\left.\frac{\partial G}{\partial z}\right|_{z=0} = -\frac{1}{4\pi\varepsilon_0}\,\frac{2h}{(s^2+h^2)^{3/2}}}$$

Sense check. A positive charge overhead pushes field lines down into the ground, so $g<0$ (downwards), as found. The factor 2 is the image: at ground level the charge and its image contribute equally to the vertical field, while their horizontal fields cancel, so the field is normal to the conductor, as it must be.

Step 4: the field directly below the cloud

Superpose the two charges:

$$E_z(s) = Q_1\,g(s;h_1) + Q_2\,g(s;h_2) = -\frac{2}{4\pi\varepsilon_0}\left[\frac{Q_1h_1}{(s^2+h_1^2)^{3/2}} + \frac{Q_2h_2}{(s^2+h_2^2)^{3/2}}\right].$$

Directly below the cloud, $s=0$ and $h/(h^2)^{3/2} = 1/h^2$, so with $2/(4\pi\varepsilon_0) = 1.798\times10^{10}\,\mathrm{m\,F^{-1}}$:

$$\begin{aligned} E_z(0) &= -\frac{2}{4\pi\varepsilon_0}\left[\frac{Q_1}{h_1^2}+\frac{Q_2}{h_2^2}\right] = -(1.798\times10^{10})\left[\frac{40}{(1.0\times10^4)^2} - \frac{40}{(6.0\times10^3)^2}\right]\mathrm{V\,m^{-1}} \\ &= -(1.798\times10^{10})\big(4.000\times10^{-7} - 1.111\times10^{-6}\big)\,\mathrm{V\,m^{-1}} = 1.28\times10^4\,\mathrm{V\,m^{-1}}. \end{aligned}$$

$12.8\,\mathrm{kV\,m^{-1}}$**, pointing upwards.** The negative charge is nearer the ground, so it wins and pulls the field upwards towards itself.

Step 5: where the ground field is zero

$E_z = 0$ when the bracket vanishes. With $Q_2 = -Q_1$, this needs

$$\frac{h_1}{(s^2+h_1^2)^{3/2}} = \frac{h_2}{(s^2+h_2^2)^{3/2}} \quad\Longleftrightarrow\quad \left(\frac{s^2+h_1^2}{s^2+h_2^2}\right)^{3/2} = \frac{h_1}{h_2}.$$

Raise both sides to the power $2/3$. With lengths in km,

$$\frac{s^2+100}{s^2+36} = \left(\frac{10}{6}\right)^{2/3} = 1.4057.$$

Multiply out and collect the s^2 terms:

$$s^2+100 = 1.4057\,s^2 + 50.61 \quad\Longrightarrow\quad s^2 = \frac{49.39}{0.4057} = 121.7\,\mathrm{km^2},$$

so $s = 11.0\,\text{km}$. By symmetry, the zero is a circle of this radius around the point below the cloud. Inside it the field points up; outside it points down (Figure 2).

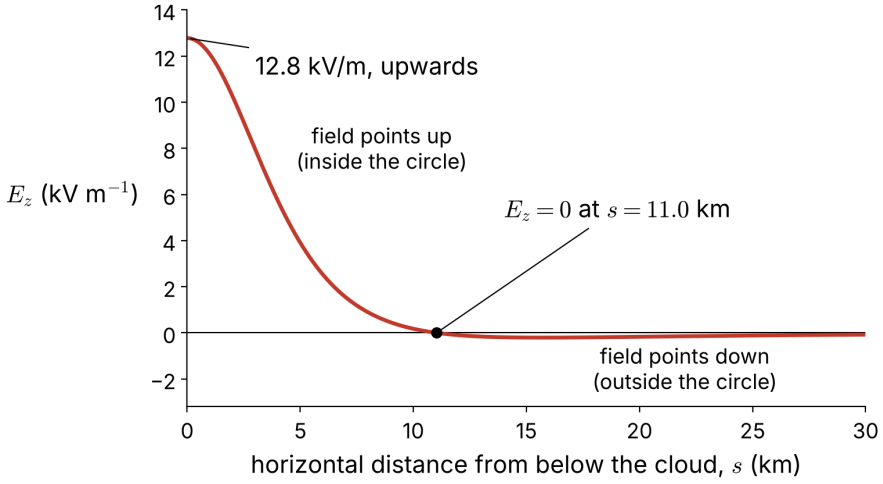


Figure 2: Vertical field at the ground against horizontal distance from the point below the cloud. Outside the zero circle the field is downwards but small: its largest magnitude there is about $0.2\,\text{kV m}^{-1}$, near $s \approx 16\,\text{km}$.

Going further: other questions answered with the same g

Because $g(s;h)$ is the ground field of a unit charge at *any* height, related questions cost one extra term each, with no new image construction.

(i) Total charge induced on the ground. At the ground, $\sigma = \varepsilon_0 E_z$, so a unit charge at height h induces $\varepsilon_0 g = -h/[2\pi(s^2+h^2)^{3/2}]$ per unit area. Over the whole ground, using rings of area $2\pi s\,\mathrm{d}s$ and the substitution $u = s^2 + h^2$,

$$\int_0^\infty \frac{-h}{2\pi(s^2+h^2)^{3/2}}\,2\pi s\,\mathrm{d}s = -h\int_0^\infty \frac{s\,\mathrm{d}s}{(s^2+h^2)^{3/2}} = -h\Big[-\frac{1}{\sqrt{s^2+h^2}}\Big]_0^\infty = -h\cdot\frac{1}{h} = -1.$$

(Every field line from the unit charge ends on the ground.) So the cloud induces a total of $-(Q_1+Q_2) = 0$: the cloud is neutral, so the ground's induced charge is positive in the inner circle and negative outside, and the two cancel exactly.

(ii) A small positive charge at the cloud base. Real clouds often also carry a small positive charge near their base. Add $Q_3 = 10\,\text{C}$ at $h_3 = 2\,\text{km}$. Directly below the cloud this adds one term, $Q_3\,g(0;h_3) = -\frac{2}{4\pi\varepsilon_0}\frac{Q_3}{h_3^2} = -(1.798\times10^{10})\,\frac{10}{(2.0\times10^3)^2} = -4.49\times10^4\,\text{V m}^{-1}$, so

$$E_z(0) = 1.28\times10^4 - 4.49\times10^4 = -3.22\times10^4\,\text{V m}^{-1}:$$

$32\,\text{kV m}^{-1}$ pointing *downwards*. A small charge close to the ground dominates, because the field of each charge at the ground falls as $1/h^2$.

Any cloud at all. For a continuous cloud charge density ρ_c, the same reasoning gives the ground field as one integral,

$$E_z(\text{ground point}) = \int g(\text{horizontal distance; height})\,\rho_c\,\mathrm{d}V',$$

with g from Step 3 and the distances measured from the ground point to each piece of cloud.

Images or Green's function: what is the difference?

Method of images used directly	Green's function
Place an image for each cloud charge, then add the four fields.	Place one image for a *unit* charge at a general height h, once.
A new cloud needs a new set of images.	A new cloud is one more term (or one integral) using the same $g(s;h)$.
Quickest for this one problem.	Pays off as soon as there is more than one problem: related questions, other clouds, induced charge.

The physics is identical: the image is built into G. The Green's function simply separates the part that depends on the *geometry* (the earthed ground: done once) from the part that depends on the *charges* (the cloud: changed at will).