Green's functions in electrostatics

An easy introduction, with two worked exam-style examples

Part IB Electromagnetism, Michaelmas Term 2026

Deepak Venkateshvaran

dv246@cam.ac.uk

Questions and corrections are welcome by email.

The whole idea in four sentences

1. Work out the potential produced by *one* unit point charge, with any earthed conductors in place. This is the **Green's function**, G.
2. A real charge distribution is just many small point charges.
3. Each small charge $\mathrm{d}q$ contributes $G\,\mathrm{d}q$ to the potential.
4. Add them up (integrate): $\phi = \int G\,\mathrm{d}q$. Done.

Supervisor's note

Where the learning is. Students already know the pieces: Coulomb's law, superposition, Gauss's law and the method of images. A Green's function simply packages them. Coulomb's $1/(4\pi\varepsilon_0 r)$ is itself a Green's function (for empty space); with conductors present, the images supply the extra terms.

Conceptually tricky. (i) Keep two points apart: the *source point* $\mathbf{r}'$, where the unit charge sits, and the *field point* $\mathbf{r}$, where the potential is measured. The integral is always over the source point. (ii) A piecewise G must be split at the field point when integrating (Example 1). (iii) Books differ by factors of ε_0 and signs (Appendix A3).

What you need to know before starting

- **Potential of a point charge** q at distance r: $\phi = q/(4\pi\varepsilon_0 r)$, where ε_0 is the permittivity of free space.
- **Superposition:** potentials of separate charges add.
- **Gauss's law at a sheet of charge:** crossing a sheet with charge σ per unit area, the electric field component normal to the sheet jumps by σ/ε_0. The potential itself does not jump.
- **Charge on a conductor surface:** $\sigma = \varepsilon_0 E_n$, where E_n is the field just outside, measured pointing away from the conductor.
- **Uniqueness:** if a potential satisfies Poisson's equation $\nabla^2\phi = -\rho/\varepsilon_0$ (with ρ the charge density) and the right values on all conductors, it is the only answer.
- **Method of images** (Sheet 1, Problems 8 and 9).

1 Start from what you already know

The potential at a point $\mathbf{r}$ due to a point charge q at $\mathbf{r}'$, in empty space, is

$$\phi(\mathbf{r}) = q \times \frac{1}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|}.$$

Read this as: (the charge) × (the potential that a *unit* charge at $\mathbf{r}'$ would produce at $\mathbf{r}$). The second factor is called the **Green's function** of empty space:

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|} \qquad (\text{units: } \mathrm{V\,C^{-1}}).$$

For a spread-out charge with density ρ, cut it into small pieces. The piece of volume $\mathrm{d}V'$ at $\mathbf{r}'$ holds charge $\mathrm{d}q = \rho(\mathbf{r}')\,\mathrm{d}V'$, and contributes $G_0\ \mathrm{d}q$. Adding all the pieces,

$$\phi(\mathbf{r}) = \int G_0(\mathbf{r}, \mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}V'.$$

You have been using this formula since Part IA. A Green's function is nothing more mysterious than "the potential of a unit point charge".

2 What changes when there are conductors

Put a point charge near an earthed metal plate. The charge attracts opposite charge onto the plate, and that *induced* charge also produces a potential. So near conductors, the potential of a unit charge is *not* G_0. We simply include the conductors' response in the definition:

Definition

$G(\mathbf{r}, \mathbf{r}')$ = the potential at $\mathbf{r}$ when a unit point charge is placed at $\mathbf{r}'$ and all the conductors are earthed.

It includes the potential of the charge that the unit charge induces on the conductors.

Then, for any charge distribution in the same region with the same earthed conductors,

$$\boxed{\phi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}V'}$$

Why this is right. Each small piece of charge produces $G\ \mathrm{d}q$, *including* the effect of the charge it induces on the conductors. Potentials add. Each piece gives zero potential on the conductors (that is how G was defined), so the total does too. By uniqueness this is the answer.

The recipe.

1. Find G: the potential of one unit charge, conductors earthed. (This is the only hard part, and it is done once for a given set of conductors.)
2. Multiply by the charge $\rho\,\mathrm{d}V'$.
3. Integrate over where the charge is.

The same idea in equation form

For a unit point charge, Poisson's equation reads

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\frac{1}{\varepsilon_0}\,\delta^3(\mathbf{r} - \mathbf{r}'), \qquad G = 0 \text{ on the earthed conductors.}$$

The **delta function** $\delta^3(\mathbf{r} - \mathbf{r}')$ is the charge density of a unit point charge: zero everywhere except at $\mathbf{r}'$, with $\int \delta^3\,\mathrm{d}V = 1$. In one dimension, $\delta(x - x')$ is the charge density of a sheet with unit charge per unit area at $x = x'$. You do not need to manipulate delta functions in this guide: the physics (Gauss's law) does the same job.

3 How to find G

(a) One dimension: straight lines with a kink

If everything depends on one coordinate x (large parallel plates, say), the "unit charge" is a flat sheet with unit charge per unit area. Then:

- away from the sheet there is no charge, so (by Gauss's law) the field is uniform and G is a **straight line**, as explained step by step in Example 1;
- G is **continuous** at the sheet;
- the **slope** of G changes at the sheet by the amount Gauss's law requires;
- $G = 0$ on the earthed plates.

Example 1 (Section 5) does exactly this.

(b) Three dimensions: the method of images

Add image charges *outside* the region so that the conductors come out at zero potential. Inside the region, the images play the role of the induced charge.

Earthed plane $z = 0$, region $z > 0$. A unit charge at $\mathbf{r}' = (x', y', z')$ needs one image, of charge -1, at the mirror point $\mathbf{r}'' = (x', y', -z')$ (Figure 1):

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}''|}\right].$$

Every point on the plane is the same distance from the charge and its image, so $G = 0$ there, as required. Example 2 (Section 6) uses this G.

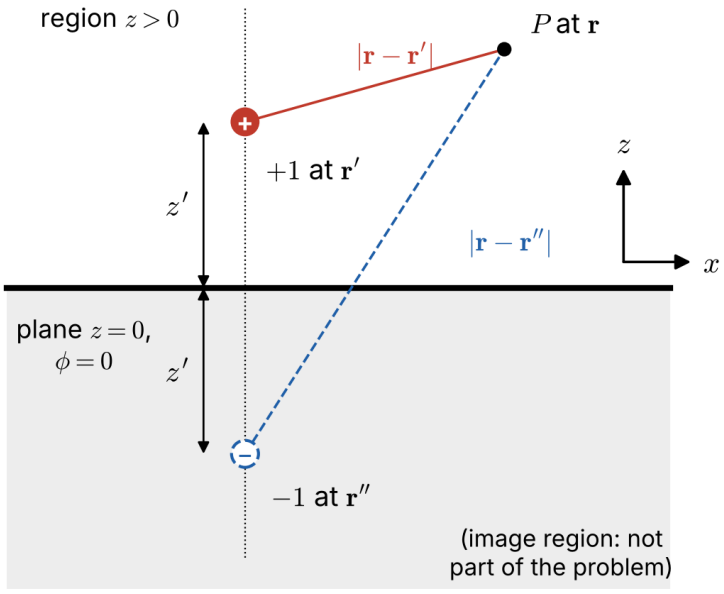


Figure 1: Green's function for the region above an earthed plane: a unit charge at $\mathbf{r}'$ plus an image of charge -1 at the mirror point $\mathbf{r}''$. Only $z > 0$ is physical; the image is a calculating device that stands in for the charge induced on the plane.

4 Reciprocity: swapping source and field point

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}).$$

In words: *a unit charge at A produces the same potential at B as a unit charge at B produces at A*, even with earthed conductors around. For empty space this is obvious, since both equal $1/(4\pi\varepsilon_0 \times \text{distance})$. For the plane it is also easy to see: the distance from $\mathbf{r}$ to the mirror image of $\mathbf{r}'$ equals the distance from $\mathbf{r}'$ to the mirror image of $\mathbf{r}$.

The same symmetry holds for whole charge distributions, and Example 2 uses it in this form:

Green's reciprocity theorem

For any two charge distributions ρ_1 and ρ_2, with potentials ϕ_1 and ϕ_2,

$$\int \rho_1 \, \phi_2 \, \mathrm{d}V = \int \rho_2 \, \phi_1 \, \mathrm{d}V.$$

Why: both sides equal $\displaystyle\iint \frac{\rho_1(\mathbf{r})\,\rho_2(\mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V\,\mathrm{d}V'$, the Coulomb interaction energy between the two distributions, which does not care which one you call "first". The theorem holds with conductors present, as long as the charges on the conductors are included in ρ_1 and ρ_2.

5 Worked example 1 (exam style): charge between earthed plates

This first example uses the simplest possible geometry: two parallel plates, so everything depends on one coordinate. Every step can be checked by intuition.

Exam-style question

Two large parallel conducting plates, at $x = 0$ and $x = L$, are both earthed. The space between them contains charge whose density depends only on x. Edge effects may be ignored.

(a) A thin sheet carrying charge $+1$ per unit area is placed in the plane $x = x'$, where $0 < x' < L$. Show that the potential between the plates is

$$G(x, x') = \begin{cases} \dfrac{x\,(L - x')}{\varepsilon_0 L}, & 0 \le x \le x', \\[2ex] \dfrac{x'\,(L - x)}{\varepsilon_0 L}, & x' \le x \le L. \end{cases}$$

Sketch G as a function of x. [5]

(b) Find the charge per unit area induced on each plate. Comment on how the induced charge is shared between the plates. [3]

(c) Explain why $\phi(x) = \int_0^L G(x, x')\,\rho(x')\,\mathrm{d}x'$ for any charge density $\rho(x)$. Use this to find $\phi(x)$ when $\rho = \rho_0$ is uniform, and check your answer directly from Poisson's equation. [5]

(d) The charge density is now $\rho = \rho_0 x/L$. Find $\phi(x)$ and the position and value of the maximum potential. Explain why the maximum is not midway between the plates. [5]

(e) Show that $G(x, x') = G(x', x)$ and state what this means physically. [2]

Final answer

(a) As given; a tent shape with peak $x'(L-x')/(\varepsilon_0 L)$ at the sheet. (b) $-(L-x')/L$ on the plate at $x=0$ and $-x'/L$ on the plate at $x=L$; total -1; the nearer plate takes more. (c) $\phi = \rho_0 x(L-x)/(2\varepsilon_0)$. (d) $\phi = \rho_0 x(L^2-x^2)/(6\varepsilon_0 L)$, maximum $\rho_0 L^2/(9\sqrt{3}\,\varepsilon_0) \approx 0.064\,\rho_0 L^2/\varepsilon_0$ at $x = L/\sqrt{3} \approx 0.58L$. (e) Swapping x and x' swaps the two lines of the formula.

(a) The Green's function

The picture first. The positive sheet sends field lines out to both sides, ending on the two earthed plates. So the potential rises from 0 at each plate to a maximum at the sheet: a tent (Figure 3). Now make it quantitative.

Step 1: why G is a straight line on each side of the sheet. This follows in three short steps (Figure 2).

(i) **The field points along x and depends only on x.** The plates and the sheet are flat and very large, so nothing changes if you slide sideways (along y or z). The field therefore cannot point sideways, and it has the same value everywhere on any plane $x =$ constant.

(ii) **The field is the same everywhere between a plate and the sheet.** Draw a Gauss box with flat faces of area A at x_1 and x_2, both on the same side of the sheet. No field crosses the box's sides (the field is along x). So the flux out of the box is $A\,E(x_2) - A\,E(x_1)$. There is no charge inside, so Gauss's law says the flux is zero:

$$A\,E(x_2) - A\,E(x_1) = 0 \quad \Longrightarrow \quad E(x_1) = E(x_2).$$

Every field line that enters the box leaves it again, so the field cannot change between x_1 and x_2. Since x_1 and x_2 are any two points on that side, the field is *uniform* there.

(iii) **A uniform field gives a straight-line potential.** $E_x = -\,\mathrm{d}G/\,\mathrm{d}x$, so a constant field means a constant slope: every step Δx along the gap changes the potential by the same amount, $|E|\,\Delta x$. A graph that rises by equal amounts for equal steps is a straight line. (It is like walking up a ramp of constant steepness.)

In equation form, (ii) and (iii) together say $\mathrm{d}^2G/\,\mathrm{d}x^2 = 0$ away from the sheet, which is Poisson's equation with no charge; its solutions are straight lines, $G =$ (constant) $\times\, x +$ (constant).

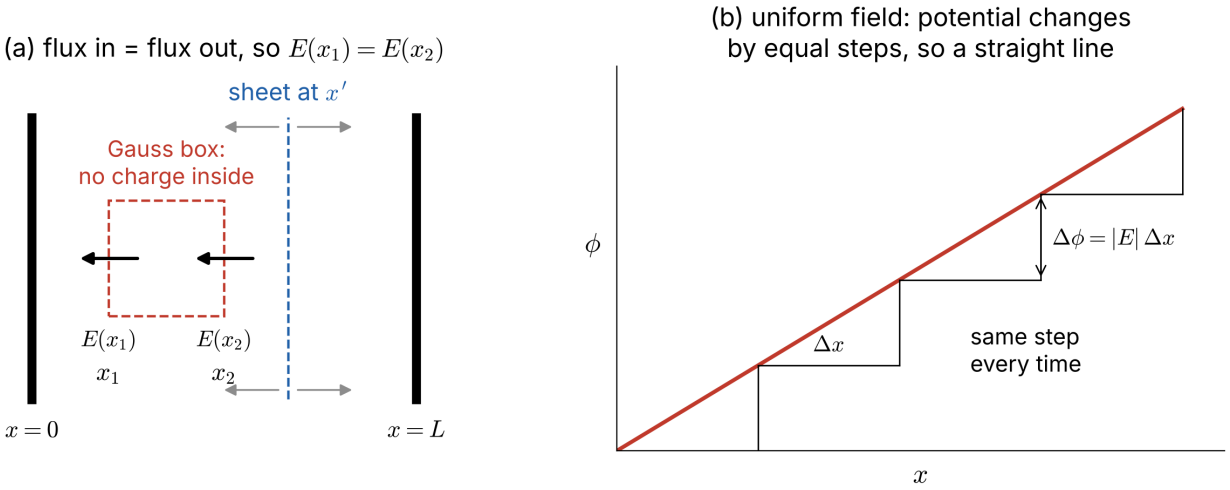


Figure 2: Why the potential is a straight line between a plate and the sheet. (a) A Gauss box between the left plate and the sheet contains no charge, so as much field leaves through its right face as enters through its left: $E(x_1) = E(x_2)$. Grey arrows show the field pointing away from the positive sheet on both sides. (b) A uniform field changes the potential by the same amount for every equal step in x, so the graph of ϕ is a straight line.

Fixing the two lines. The left line must pass through $G = 0$ at $x = 0$ (earthed plate), so it is Ax. The right line must pass through $G = 0$ at $x = L$, so it is $B(L - x)$:

$$G = Ax \quad (x < x'), \qquad G = B(L - x) \quad (x > x'),$$

with the slopes A and B still to find. Two more facts fix them.

Step 2: no jump in potential at the sheet.

$$Ax' = B(L - x'). \tag{1}$$

Step 3: Gauss's law at the sheet. The field is $E_x = -\,\mathrm{d}G/\,\mathrm{d}x$. On the left it is $-A$ (pointing left, towards the left plate); on the right it is $+B$ (pointing right). Crossing the sheet, E_x jumps by $\sigma/\varepsilon_0 = 1/\varepsilon_0$:

$$B - (-A) = \frac{1}{\varepsilon_0} \quad \Longrightarrow \quad A + B = \frac{1}{\varepsilon_0}. \tag{2}$$

Step 4: solve. From (1), $B = Ax'/(L - x')$. Put this into (2): $A\left[1 + x'/(L - x')\right] = A\,L/(L - x') = 1/\varepsilon_0$. So

$$A = \frac{L - x'}{\varepsilon_0 L}, \qquad B = \frac{x'}{\varepsilon_0 L},$$

which gives the stated G.

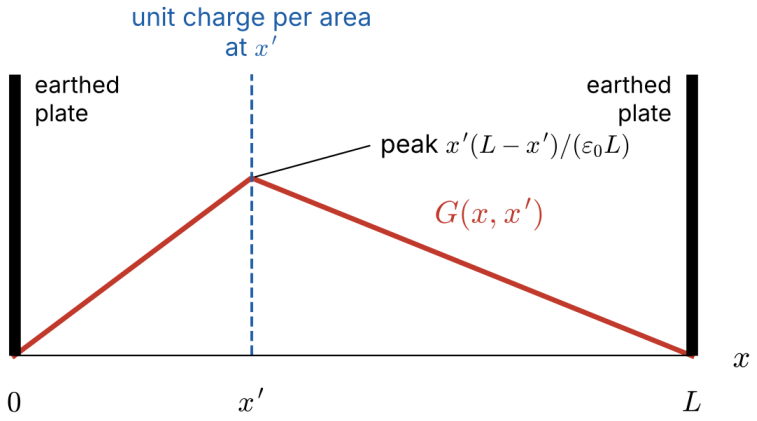


Figure 3: The Green's function between earthed plates: the potential of a sheet of unit charge per unit area at x'. It is a straight line on each side, continuous at the sheet, with a kink whose size is set by Gauss's law.

Sense check. The peak height $x'(L-x')/(\varepsilon_0 L)$ is largest, $L/(4\varepsilon_0)$, when the sheet is midway, and falls to zero as the sheet approaches either plate. That makes sense: a charge pressed against an earthed plate is cancelled by its induced charge.

(b) Induced charge on the plates

Use $\sigma = \varepsilon_0 E_n$, with E_n the field pointing away from the plate into the gap.

- **Plate at $x = 0$:** "away" is $+x$, and $E_x = -A$, so $\sigma_0 = -\varepsilon_0 A = -\dfrac{L-x'}{L}$.
- **Plate at $x = L$:** "away" is $-x$, and $E_x = +B$, so $\sigma_L = -\varepsilon_0 B = -\dfrac{x'}{L}$.

Check: $\sigma_0 + \sigma_L = -1$. Every field line from the unit sheet ends on a plate, so the plates carry exactly the opposite charge.

Comment. The nearer plate gets the larger share. A sheet at $x' = L/4$ induces $-\frac{3}{4}$ on the near plate and $-\frac{1}{4}$ on the far one. Reason: the sheet is at the same potential above both plates, and the shorter gap needs a stronger field to produce the same potential drop, so it carries more field lines.

(c) Any charge distribution; the uniform case

Why the integral works. Slice the charge into thin sheets. The slice from x' to $x' + \mathrm{d}x'$ carries $\rho(x')\,\mathrm{d}x'$ per unit area, so by part (a) it adds $G(x,x')\,\rho(x')\,\mathrm{d}x'$ to the potential at x. Potentials add, and every contribution is zero on both plates. Therefore $\phi(x) = \int_0^L G(x,x')\rho(x')\,\mathrm{d}x'$.

Splitting the integral (the one place to be careful). Fix the field point x. Slices to its *left* ($x' < x$) have x on their right-hand side, so use the second line of G, namely $x'(L-x)/(\varepsilon_0 L)$. Slices to its *right* ($x' > x$) use the first line, $x(L-x')/(\varepsilon_0 L)$:

$$\phi(x) = \frac{1}{\varepsilon_0 L}\left[(L-x)\int_0^x x'\rho(x')\,\mathrm{d}x' \;+\; x\int_x^L (L-x')\rho(x')\,\mathrm{d}x'\right]. \tag{3}$$

Inside each integral x is a constant; only x' varies.

Uniform charge, $\rho = \rho_0$.

$$\int_0^x x'\,\mathrm{d}x' = \frac{x^2}{2}, \qquad \int_x^L (L-x')\,\mathrm{d}x' = \frac{(L-x)^2}{2},$$

$$\phi(x) = \frac{\rho_0}{\varepsilon_0 L}\left[\frac{(L-x)x^2}{2} + \frac{x(L-x)^2}{2}\right] = \frac{\rho_0\, x(L-x)}{2\varepsilon_0 L}\big[x + (L-x)\big] = \frac{\rho_0\, x(L-x)}{2\varepsilon_0}.$$

Direct check. $\mathrm{d}^2\phi/\,\mathrm{d}x^2 = -\rho_0/\varepsilon_0$ (Poisson's equation) and $\phi(0) = \phi(L) = 0$. By uniqueness the two methods must agree, and they do. The potential is a symmetric parabola peaking midway, at $\rho_0 L^2/(8\varepsilon_0)$ (Figure 4).

(d) Charge density increasing across the gap

Now $\rho(x') = \rho_0 x'/L$. The two integrals in (3) are

$$\int_0^x x' \cdot \frac{\rho_0 x'}{L}\,\mathrm{d}x' = \frac{\rho_0}{L}\frac{x^3}{3}, \qquad \int_x^L (L-x')\frac{\rho_0 x'}{L}\,\mathrm{d}x' = \frac{\rho_0}{L}\left[\frac{Lx'^2}{2} - \frac{x'^3}{3}\right]_x^L = \frac{\rho_0}{L}\left[\frac{L^3}{6} - \frac{Lx^2}{2} + \frac{x^3}{3}\right].$$

Put these into (3):

$$\phi(x) = \frac{\rho_0}{\varepsilon_0 L^2}\left[\frac{Lx^3}{3} - \frac{x^4}{3} + \frac{L^3 x}{6} - \frac{Lx^3}{2} + \frac{x^4}{3}\right].$$

The x^4 terms cancel and $Lx^3/3 - Lx^3/2 = -Lx^3/6$, leaving

$$\boxed{\phi(x) = \frac{\rho_0}{6\varepsilon_0 L}\, x\,(L^2 - x^2)}$$

Check. $\phi(0) = \phi(L) = 0$, and $\mathrm{d}^2\phi/\,\mathrm{d}x^2 = \frac{\rho_0}{6\varepsilon_0 L}(-6x) = -\rho/\varepsilon_0$.

Maximum. $\mathrm{d}\phi/\,\mathrm{d}x = \frac{\rho_0}{6\varepsilon_0 L}(L^2 - 3x^2) = 0$ at $x = L/\sqrt{3} \approx 0.58L$, where

$$\phi_{\max} = \frac{\rho_0}{6\varepsilon_0 L}\cdot\frac{L}{\sqrt{3}}\cdot\frac{2L^2}{3} = \frac{\rho_0 L^2}{9\sqrt{3}\,\varepsilon_0} \approx 0.064\,\frac{\rho_0 L^2}{\varepsilon_0}.$$

Why not midway? At the maximum, $E_x = 0$: it is the dividing line between field lines that go left and field lines that go right. More of the charge sits near the right-hand plate, so the dividing line moves towards it. **Size check:** the total charge is $\rho_0 L/2$ per unit area, half that of the uniform case, and $\phi_{\max}$ is about half the uniform value of $0.125\,\rho_0 L^2/\varepsilon_0$.

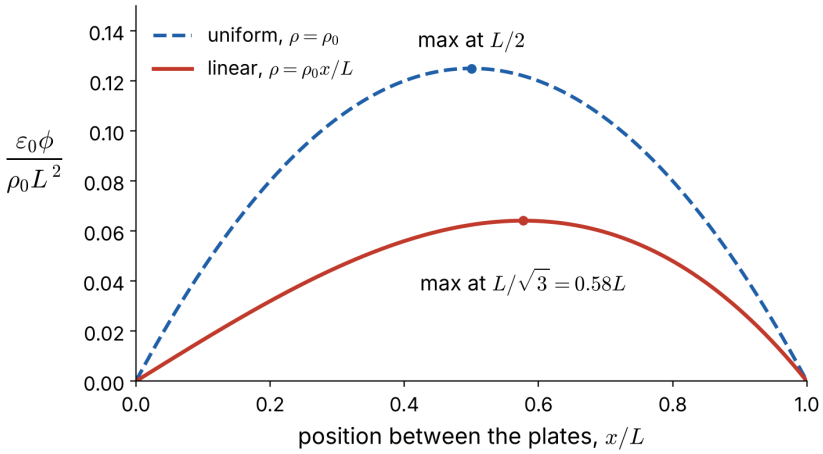


Figure 4: Potential between the earthed plates for uniform charge (dashed) and for charge increasing linearly across the gap (solid). The maximum moves towards the side with more charge.

(e) Reciprocity

Take $x < x'$. $G(x, x')$ is given by the first line: $x(L - x')/(\varepsilon_0 L)$. For $G(x', x)$ the roles swap: the field point is now x', to the right of the source at x, so use the second line with the letters exchanged: $x(L - x')/(\varepsilon_0 L)$. They are equal.

Physically: a unit sheet at A gives the same potential at B as a unit sheet at B gives at A (Section 4). Hence the compact form $G = x_<(L - x_>)/(\varepsilon_0 L)$, where $x_<$ and $x_>$ are the smaller and larger of x and x'.

What example 1 teaches

- G came from two physical facts only: the potential is continuous at a sheet, and Gauss's law fixes the jump in field.
- Once G is known, every charge distribution between the same plates is one integral; parts (c) and (d) reuse the same G.
- The only care needed is splitting the integral at the field point.

6 Worked example 2 (exam style): a disc held at a fixed potential in an earthed plane

In this example there is no free charge at all. Instead, part of the earthed plane is held at a fixed potential. The Green's function tells us how much each part of the plane "matters" for the potential at a chosen point, and the answer has a simple picture: *the potential at a point is the average of the plane's potential over the view from that point.*

Exam-style question

A large flat conducting plane, $z = 0$, is earthed except for a circular disc of radius a centred on the origin. The disc is insulated from the rest of the plane by a very thin gap and is held at potential V_0. There is no charge in the region $z > 0$.

(a) A point charge q is placed at height z above the plane, which is now earthed everywhere. Using the method of images, show that the charge per unit area induced on the plane at horizontal distance R from the foot of the charge is

$$\sigma_q(R) = -\frac{q\,z}{2\pi\,(R^2+z^2)^{3/2}}.$$

[4]

(b) Using Green's reciprocity theorem, $\int \rho_1\phi_2\,\mathrm{d}V = \int \rho_2\phi_1\,\mathrm{d}V$, show that in the original problem the potential at a point P at height z is

$$\phi(P) = \int_{\text{plane}} \phi_S\, W\,\mathrm{d}S, \qquad W = \frac{z}{2\pi\,(R^2+z^2)^{3/2}},$$

where ϕ_S is the potential on the plane and R is the horizontal distance from P. [5]

(c) Show that $\int W\,\mathrm{d}S = 1$ over the whole plane, and explain what this means. [3]

(d) Find the potential on the axis of the disc, and the height at which it equals $V_0/2$. [5]

(e) Show that far above the disc, on its axis, $\phi \approx V_0 a^2/(2z^2)$, and explain this result without the exact formula. [3]

Final answer

(a) As given. (b) As given: $W = -\sigma_q/q$. (c) The weights add up to 1, so the potential is a weighted average of the potential on the plane; here $0 \le \phi \le V_0$. (d) $\phi = V_0\left(1 - \dfrac{z}{\sqrt{z^2+a^2}}\right) = V_0(1-\cos\alpha)$, where α is the angle at which P sees the edge of the disc; $\phi = V_0/2$ at $z = a/\sqrt{3} \approx 0.58a$. (e) $\phi \approx V_0 a^2/(2z^2)$: from far away the disc fills a solid angle $\pi a^2/z^2$ out of the 2π that P sees.

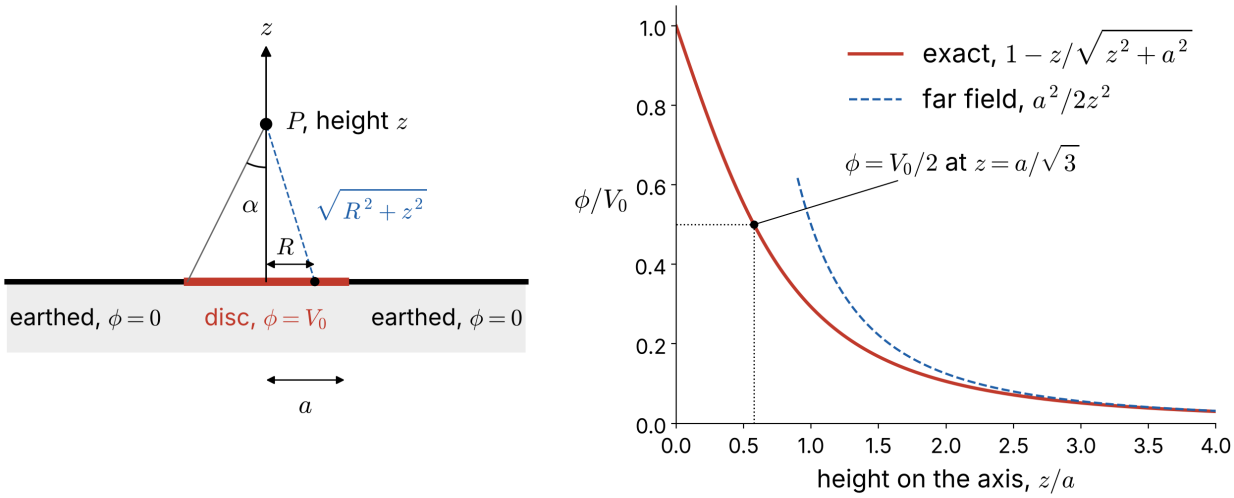


Figure 5: Left: the disc of radius a at potential V_0, insulated from the surrounding earthed plane by a gap too thin to show. P is on the axis at height z; R is the horizontal distance of a point on the plane from P; α is the angle at which P sees the edge of the disc. Right: the potential on the axis (solid) and its far-field form (dashed).

(a) Charge induced by a point charge (this is the Green's function at work)

With the plane earthed, the potential of the point charge is $q\,G$, where G is the half-space Green's function of Section 3(b): the charge q at height z plus an image $-q$ at depth z below the plane.

Take a point on the plane at horizontal distance R from the foot of the charge. Both the charge and the image are a distance $d = \sqrt{R^2 + z^2}$ away.

- The real charge $+q$ is *above* the point and pushes away from itself, so its field there has a downward component of size $\dfrac{q}{4\pi\varepsilon_0 d^2} \times \dfrac{z}{d}$ (the factor z/d picks out the vertical part).
- The image $-q$ is *below* and pulls towards itself, so its field is also downward, with the same size.

The horizontal parts cancel; the vertical parts add:

$$E_z = -\frac{2qz}{4\pi\varepsilon_0 d^3}.$$

The field just above the plane, pointing away from the conductor, is E_z, so $\sigma_q = \varepsilon_0 E_z$:

$$\sigma_q(R) = -\frac{q\,z}{2\pi(R^2 + z^2)^{3/2}}.$$

This is the result of Sheet 1, Problem 8 (the thundercloud). It is negative, as expected for charge induced by a positive charge, and largest directly below the charge.

(b) Reciprocity turns this into the answer

Compare two situations that share the same plane.

- **Situation 1 (the real problem):** disc at V_0, rest of the plane earthed, no other charges. Its potential is ϕ_1. All its charges sit on the plane (we never need to find them).
- **Situation 2 (the helper problem):** the whole plane earthed, and a point charge q at P. Its potential is $\phi_2 = qG$. Its charges are q at P plus σ_q on the plane, from part (a).

Apply the reciprocity theorem (Section 4), $\int \rho_1\phi_2\,\mathrm{d}V = \int \rho_2\phi_1\,\mathrm{d}V$.

Left-hand side. The charges of situation 1 are all on the plane, where $\phi_2 = 0$ (the plane is earthed in situation 2). So the left-hand side is *zero*.

Right-hand side. The charges of situation 2 are q at P, where $\phi_1 = \phi(P)$, and σ_q on the plane, where $\phi_1 = \phi_S$:

$$0 = q\,\phi(P) + \int \sigma_q\,\phi_S\,\mathrm{d}S \qquad \Longrightarrow \qquad \phi(P) = \int \phi_S \left(-\frac{\sigma_q}{q}\right)\mathrm{d}S = \int \phi_S\,W\,\mathrm{d}S,$$

with $W = -\sigma_q/q = z/[2\pi(R^2+z^2)^{3/2}]$, as required.

In words: the parts of the plane where a charge at P would induce the most charge are the parts that most affect the potential at P. Those are the parts nearest P.

(c) The weights add up to one

Use rings centred below P: a ring of radius R and width $\mathrm{d}R$ has area $2\pi R\,\mathrm{d}R$. So

$$\int W\,\mathrm{d}S = \int_0^\infty \frac{z}{2\pi(R^2+z^2)^{3/2}}\,2\pi R\,\mathrm{d}R = z\int_0^\infty \frac{R\,\mathrm{d}R}{(R^2+z^2)^{3/2}} = z\Big[-\frac{1}{\sqrt{R^2+z^2}}\Big]_0^\infty = z\cdot\frac{1}{z} = 1.$$

(The integral is done with the substitution $u = R^2+z^2$, $\mathrm{d}u = 2R\,\mathrm{d}R$.) Equivalently, the total induced charge is $-q$: every field line from q ends on the plane.

What it means. $\phi(P)$ is a *weighted average* of the potential on the plane, with positive weights adding up to 1. So:

- if the whole plane were at V_0, then $\phi = V_0$ everywhere (correct: a constant potential has no field and satisfies everything);
- in our problem, $0 \le \phi \le V_0$ everywhere above the plane. The potential can never overshoot the values on the boundary.

The picture: the weight is a solid angle. Seen from P, a small patch $\mathrm{d}S$ of the plane at distance d fills a solid angle $\mathrm{d}\Omega = \mathrm{d}S\cos\theta/d^2$, where θ is the angle between the line of sight and the vertical, so $\cos\theta = z/d$. Hence $\mathrm{d}\Omega = z\,\mathrm{d}S/d^3$ and

$$W\,\mathrm{d}S = \frac{\mathrm{d}\Omega}{2\pi}, \qquad \text{so} \qquad \boxed{\phi(P) = \frac{1}{2\pi}\int \phi_S\,\mathrm{d}\Omega}$$

Looking down from P you see a hemisphere's worth of plane (solid angle 2π). *The potential at P is the plane's potential averaged over what P sees.* Patches close to P fill more of the view, so they count more.

(d) The potential on the axis

On the axis, R is just the distance from the origin. Only the disc ($R < a$) is at nonzero potential, so

$$\phi(0,0,z) = V_0\int_0^a \frac{z}{2\pi(R^2+z^2)^{3/2}}\,2\pi R\,\mathrm{d}R = V_0\,z\Big[-\frac{1}{\sqrt{R^2+z^2}}\Big]_0^a = V_0\left(1-\frac{z}{\sqrt{z^2+a^2}}\right).$$

Same answer from the picture. From P, the edge of the disc is seen at angle α from the vertical, with $\cos\alpha = z/\sqrt{z^2+a^2}$ (Figure 5). The disc fills a cone of half-angle α, whose solid angle is $2\pi(1-\cos\alpha)$. The average over the view is therefore

$$\boxed{\phi = V_0\,\frac{2\pi(1-\cos\alpha)}{2\pi} = V_0(1-\cos\alpha) = V_0\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)}$$

Checks.

- Just above the disc ($z \to 0$): $\phi \to V_0$. The disc fills the whole view.
- Very large disc ($a \to \infty$): $\phi \to V_0$. The whole plane is at V_0.
- Far away ($z \to \infty$): $\phi \to 0$. The disc becomes a dot in the view.

Half potential. $\phi = V_0/2$ needs $\cos\alpha = \frac{1}{2}$, i.e. $\alpha = 60°$. Then $z/\sqrt{z^2+a^2} = \frac{1}{2}$, so $4z^2 = z^2 + a^2$ and

$$z = \frac{a}{\sqrt{3}} \approx 0.58\,a.$$

At this height the disc fills exactly half of what P sees.

(e) Far above the disc

From the formula. For $z \gg a$, use $(1+u)^{-1/2} \approx 1 - \frac{1}{2}u$ with $u = a^2/z^2$:

$$\frac{z}{\sqrt{z^2+a^2}} = \left(1 + \frac{a^2}{z^2}\right)^{-1/2} \approx 1 - \frac{a^2}{2z^2} \qquad \Longrightarrow \qquad \phi \approx \frac{V_0 a^2}{2z^2}.$$

From the picture. Far away, the disc looks small: its solid angle is its area divided by the distance squared, $\Omega \approx \pi a^2/z^2$. Everything else in view is at zero potential. So

$$\phi \approx V_0\,\frac{\Omega}{2\pi} = V_0\,\frac{\pi a^2/z^2}{2\pi} = \frac{V_0 a^2}{2z^2}.$$

The potential falls as $1/z^2$, like a dipole's rather than a point charge's, because the disc's charge is accompanied by an equal and opposite induced charge on the earthed plane right next to it.

What example 2 teaches

- The image construction gives G; the charge that a unit charge induces on the plane is the weighting function for *any* potential on the plane.
- The weight is a solid angle: *the potential at a point is the boundary potential averaged over the view from that point*. This makes every limit obvious.
- Reciprocity swaps a hard problem (find ϕ everywhere from a given boundary potential) for an easy one (a point charge above an earthed plane).

7 Are Green's functions guessed?

Sometimes, and that is allowed. Sometimes they are derived systematically.

Why a guess is allowed. By the uniqueness theorem, there is only one function that (i) has a unit charge at $\mathbf{r}'$ and no other charge in the region, and (ii) is zero on the earthed conductors. So if you guess a G and it passes both checks, it *is* the Green's function. How you found it does not matter.

The two examples in this guide.

- **The plane (Example 2) is an inspired guess.** The method of images guesses that one mirror charge -1 will make the plane zero, checks that it does, and lets uniqueness do the rest. The guess works because of the plane's mirror symmetry.
- **The plates (Example 1) are not guessed.** Every step was forced: Gauss's law gives straight lines, earthing fixes where they start, and the jump in field at the sheet fixes the slopes.

The limitation of guessing. Images work only for a few shapes: a plane, a sphere, a wedge whose angle divides 180° exactly (Sheet 1, Problem 9) and some cylinder problems. For most shapes no set of image charges exists. The general methods, met in Part IB Mathematical Methods, are:

Method	When it works
Images (a guess, checked)	Planes, spheres, some wedges and cylinders
Solve and match	Problems that depend on one coordinate (Example 1)
Expand in standing waves	Boxes, spheres, cylinders: any shape with simple boundaries (below)
Fourier transform	Regions with no boundaries
Numerical solution	Any shape (engineering software does this)

Example 1 again, with no guessing at all: standing waves

This method needs no physical insight, and it gives the same tent as Example 1. It works like a guitar string.

Step 1: building blocks. Any shape that is zero at both plates ($x = 0$ and $x = L$) can be built from the standing waves of a string fixed at both ends:

$$s_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots$$

(This is the Fourier sine series.) So write $G(x, x') = \sum_n c_n\, s_n(x)$, with coefficients c_n to find. Each building block is already zero on both plates, so the boundary condition is satisfied automatically.

Step 2: each building block is easy to differentiate.

$$\frac{\mathrm{d}^2 s_n}{\mathrm{d}x^2} = -\Big(\frac{n\pi}{L}\Big)^2 s_n.$$

Step 3: write the unit sheet in the same building blocks. The charge density of the sheet, $\delta(x - x')$, is zero except at x'. Its sine-series coefficients are found in the usual way, $\frac{2}{L}\int_0^L \delta(x - x')\, s_n(x)\,\mathrm{d}x = \frac{2}{L}\, s_n(x')$, so

$$\delta(x - x') = \frac{2}{L}\sum_n s_n(x')\, s_n(x).$$

In words: the sheet is made of each standing wave in proportion to how large that wave is at the sheet's position.

Step 4: match term by term. Poisson's equation, $\mathrm{d}^2G/\,\mathrm{d}x^2 = -\delta(x - x')/\varepsilon_0$, must hold for each building block separately:

$$-\Big(\frac{n\pi}{L}\Big)^2 c_n = -\frac{2}{\varepsilon_0 L}\, s_n(x') \quad\Longrightarrow\quad c_n = \frac{2L}{\varepsilon_0 \pi^2 n^2}\, s_n(x').$$

So

$$\boxed{G(x, x') = \frac{2L}{\varepsilon_0 \pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{L}\sin\frac{n\pi x'}{L}}$$

Is it the same tent? Figure 6 adds up the first few terms. The first term alone is a smooth hump; with 20 terms the sum is almost exactly the tent of Example 1, kink and all. A quick exact check:

put the sheet in the middle and look at the middle, $x = x' = L/2$. Then $\sin^2(n\pi/2)$ is 1 for odd n and 0 for even n, and using the known sum $1 + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{8}$,

$$G\left(\tfrac{L}{2}, \tfrac{L}{2}\right) = \frac{2L}{\varepsilon_0 \pi^2} \cdot \frac{\pi^2}{8} = \frac{L}{4\varepsilon_0},$$

exactly the peak height found in Example 1. (Uniqueness guarantees that the two methods agree everywhere, not just here.)

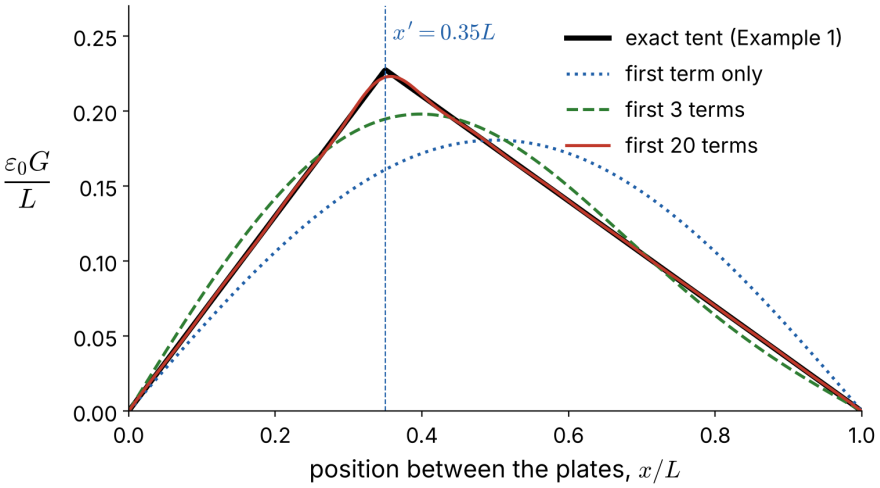


Figure 6: The standing-wave series for the Green's function between earthed plates, with the sheet at $x' = 0.35L$. One term gives a smooth hump; three terms already lean towards the sheet; twenty terms are almost indistinguishable from the exact tent of Example 1.

Which method to use? The series is systematic but converges slowly near the kink; the straight-line method of Example 1 gives the closed form directly. In practice, use symmetry and physical insight (images, Gauss's law) when the shape allows it, and standing waves when it does not. Notice also that the series is visibly unchanged when x and x' are swapped: reciprocity again.

8 Summary

What G is	The potential at $\mathbf{r}$ of a unit charge at $\mathbf{r}'$, with the conductors earthed
Empty space	$G_0 = 1/(4\pi\varepsilon_0\lvert\mathbf{r} - \mathbf{r}'\rvert)$, i.e. Coulomb
Using it	$\phi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\,\mathrm{d}V'$
Finding it	1D: straight lines, continuous, kink set by Gauss's law. 3D: images.
Reciprocity	$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$
Guessed or derived?	Images: a guess, legitimate because of uniqueness. Otherwise systematic: solve and match, or standing waves
Plane at a given potential	$\phi(P) = \frac{1}{2\pi}\int \phi_S\,\mathrm{d}\Omega$: average over the view from P

Common pitfalls

- Mixing up the source point $\mathbf{r}'$ (integrated over) and the field point $\mathbf{r}$ (held fixed).

- Using one branch of a piecewise G for the whole integral instead of splitting at the field point.
- Adding the induced charge again when it is already built into G.
- Comparing formulas from different books without checking the convention (Appendix A3).

Extensions

1. In Example 1, hold the plate at $x = L$ at potential V_0 instead of earth. Show that the answer is the earthed-plate answer plus $V_0 x/L$, and explain why the two simply add.
2. In Example 2, find the potential on the axis when the plane is earthed except for a ring $a < R < b$ held at V_0. *(Hint: the ring fills the solid angle between two cones.)*
3. Find the Green's function for a line charge (unit charge per unit length) between two coaxial earthed cylinders of radii a and b. *(Hint: on each side of the line charge the potential is $A + B \ln s$, where s is the distance from the axis.)*

Appendix: for the curious

A1 Checking that Coulomb's law satisfies the delta-function equation

Away from $\mathbf{r}'$, $\nabla^2(1/|\mathbf{r} - \mathbf{r}'|) = 0$. To see the delta function at $\mathbf{r}'$, integrate $\nabla^2 G_0$ over a small sphere of radius a around $\mathbf{r}'$ and use the divergence theorem:

$$\int \nabla^2 G_0 \, \mathrm{d}V = \oint \nabla G_0 \cdot \, \mathrm{d}\mathbf{S} = \left(-\frac{1}{4\pi\varepsilon_0 a^2} \right) \times 4\pi a^2 = -\frac{1}{\varepsilon_0},$$

which is what $-\delta^3/\varepsilon_0$ integrates to. This is just Gauss's law for a unit charge.

A2 The general boundary formula

For any two functions u and v, $\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$. Integrating over the region $\mathcal{V}$ and using the divergence theorem gives **Green's second identity**,

$$\int_{\mathcal{V}} (u\nabla^2 v - v\nabla^2 u) \, \mathrm{d}V = \oint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}S,$$

where S is the boundary of $\mathcal{V}$ and $\partial/\partial n$ is the derivative along its outward normal. With $u = \phi$ and $v = G$ (taking $\mathbf{r}'$ as the variable), it gives

$$\phi(\mathbf{r}) = \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}') \, \mathrm{d}V' - \varepsilon_0 \oint_S \phi_S(\mathbf{r}') \, \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} \, \mathrm{d}S'.$$

For the plane, $-\varepsilon_0 \, \partial G/\partial n'$ is exactly the weighting function W of Example 2, so this reproduces part (b) by a more mathematical route. With $u = G(\mathbf{r}, \mathbf{r}_A)$ and $v = G(\mathbf{r}, \mathbf{r}_B)$, the same identity proves $G(\mathbf{r}_A, \mathbf{r}_B) = G(\mathbf{r}_B, \mathbf{r}_A)$ for conductors of any shape.

A3 A note on conventions

Here G is the potential of a unit charge, so $\nabla^2 G = -\delta^3/\varepsilon_0$. Mathematical Methods texts usually write $\nabla^2 \mathcal{G} = \delta^3$, which gives $\mathcal{G} = -1/(4\pi|\mathbf{r} - \mathbf{r}'|) = -\varepsilon_0 G$. The physics is identical; only the bookkeeping differs. Always check which equation a quoted Green's function satisfies.